Numbers: Abundant, Deficient, Perfect, and Amicable

What are "abundant" numbers? Why would such numbers be called plentiful? "Abundant" is a strange way to describe a number, and equally strange are descriptions such as "deficient," "perfect," and "amicable." But these descriptions of numbers came about because the ancient Greek mathematicians were intrigued by certain characteristics of positive **integers**.

The Greeks discovered, for example, that some numbers are equal to the sum of their divisors; for instance, 6 is equal to the sum of its proper divisors 3, 2, and 1. (Although 6 is a divisor of 6, it is not considered a "proper" divisor.) Greek mathematicians discovered a sense of balance or perfection in such numbers, and labeled them "perfect."

As an extension of the idea of perfect numbers, the concept of "abundant" and "deficient" numbers emerged. If the sum of the proper divisors of a number is greater than the number itself, then the number is called abundant or excessive. The proper divisors of 12 are 1, 2, 3, 4, and 6. Because the sum of its proper divisors (1 + 2 + 3 + 4 + 6 = 16) is greater than 12, 12 is an abundant number. Numbers like 8, whose proper divisors have a sum that is less than the number itself, are called deficient or defective.

The Greeks, who regarded the proper divisors of a number to be the number's "parts," were the first to refer to perfect numbers—numbers that are the exact sum of their parts. Later, the philosopher Nicomachus,* in his *Introduction to Arithmetic,* would coin the terms "abundant" and "deficient," attaching moral qualities to these numbers. From a mathematical point of view, abundant, deficient, and perfect numbers are "abundant," "deficient," and "perfect" only in how the sum of their proper divisors compares to the numbers themselves.

*__Nicomachus' treatise was the first work to treat arithmetic independent of geometry.__

Abundant Numbers

Twelve is the first abundant number. The next abundant number is 18 because the proper divisors sum to 21 (1 + 2 + 3 + 6 + 9). The first five abundant numbers are 12, 18, 20, 24, and 30. As it turns out, the twenty-one abundant numbers under 100 are all even. Not all abundant numbers, however, are even; the first odd abundant number is 945. Every multiple of an abundant number is itself abundant, so there is an infinite number of abundant numbers. In 1998, the mathematician Marc Deleglise showed that roughly one-quarter of all the positive integers are abundant.

Deficient Numbers

Deficient numbers occur more frequently than abundant numbers. In other words, the sum of the proper divisors of most numbers is less than the numbers themselves. Examples of deficient numbers include 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, and 23.

Perfect Numbers

Perfect numbers, which occur infrequently, are the most interesting of the three types. The ancient Greeks apparently knew the first four: 6, 28, 496, and 8,128. The fifth perfect number, however, is 33,550,336, and the sixth is 8,589,869,056. As of June 1999, the thirty-eighth and largest known perfect number is $2^{6972592}(2^{6972593}- 1)$, a number roughly 4 million digits long!

Perfect numbers have some interesting properties. For example, the sum of the reciprocals of the divisors of a perfect number is always equal to 2. Consider the sum of the reciprocal of the divisors of 6 (1, 2, 3, and 6) and 28 (1, 2, 4, 7, 14, and 28).

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{7} + \frac{1}{14} + \frac{1}{28} = 2$$

The Historic Search. The Pythagoreans encountered the oldest problem in number theory—the problem of finding all of the perfect numbers. Euclid was the first to produce a mathematical result concerning perfect numbers. In 300 B.C.E., he proved in his book *Elements* that any number of the form $2^{n-1}(2^n- 1)$ is a perfect number whenever $2^n- 1$ is a prime number (a number that has no proper divisors except 1). In 1757, the Swiss mathematician Euler made further progress by proving the converse of this statement: every even perfect number is of the form $2^{n-1}(2^n 1)$, where 2^n1 is a prime number.

Primes of the form $2^n- 1$ are mathematically interesting in their own right; they are known as "Mersenne primes," named after the French monk Marin Mersenne. Because of the results of Euclid and Euler, every Mersenne prime corresponds to a perfect number and vice versa. Thus, the problem of finding all the even perfect numbers is reduced to the problem of finding all the Mersenne primes.

Mathematicians have conjectured, but not proven, that there is an infinite number of Mersenne primes and hence an infinite amount of even perfect numbers. It is not known if there are any odd perfect numbers. The largest perfect number currently known, $2^{6972592}(2^{6972593}- 1)$, was discovered by using computer programs, and this number corresponds to the thirty-eighth known Mersenne prime.

Amicable Numbers

Related to abundant, deficient, and perfect numbers are *amicable numbers*. Amicable numbers are described in pairs. Two numbers are amicable if the sum of the proper divisors of each number equals the other number in the pair. The first pair of amicable numbers is 220 and 284. The sum of the proper divisors of 220 is 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284, whereas the sum of the proper divisors of 284 is 1 + 2 + 4 + 71 + 142 = 220.

Amicable numbers occur infrequently; there are only a total of 350 amicable pairs whose smaller number has less than ten digits. The amicable numbers were also first noted by the Pythagoreans, who would be the first, among many, to attach mystical qualities to these "friendly" numbers.

There currently are more than 7,500 known pairs of amicable numbers. There is no mathematical formula describing the form of all of the amicable numbers. As with perfect numbers, today's searches for amicable pairs continue on computers. It is conjectured that the number of amicable pairs is infinite.

Those Evasive Perfect Numbers

Despite being tackled by such great mathematicians as Pierre de Fermat and Leonhard Euler, the problem of finding all the perfect numbers remains unsolved today. Although mathematicians have been unable to solve this problem, their attempts to do so have resulted in many advances in number theory.